# Geodesics on extensions of Lie groups and stability; the superconductivity equation

Cornelia Vizman

*West University of Timisoara, Department of Mathematics*
*Bd. V.Parvan 4, 1900-Timisoara, Romania*
*e-mail: vizman@math.uvt.ro*

**Abstract**

The equations of motion of an ideal charged fluid, respectively the superconductivity equation (both in a given magnetic field) are showed to be geodesic equations of a general, respectively a central extension of the group of volume preserving diffeomorphisms with right invariant metrics. For this, quantization of the magnetic flux is required. We do curvature computations in both cases in order to get informations about the stability.

*Keywords*: superconductivity equation, geodesic, curvature, stability.
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## 1 Introduction

Important partial differential equations were obtained as geodesic equations on diffeomorphism groups with right invariant $L^2$ or $H^1$ metrics: Euler equation of motion of an incompressible ideal fluid is the geodesic equation on the volume preserving diffeomorphism group with $L^2$ metric [1] [2], Burger's equation is the geodesic equation on $\mathrm{Diff}(S^1)$ with $L^2$ metric [3], the Korteweg-de-Vries equation is the geodesic equation on the Virasoro-Bott group with $L^2$ metric [4], the Camassa-Holm shallow water equation [5] is the geodesic equation on $\mathrm{Diff}(S^1)$ and on the Virasoro-Bott group with $H^1$ metrics [6] [7] [8], the averaged Euler equation is the geodesic equation on the volume preserving diffeomorphism group with $H^1$ metric [9] [10], the equations of ideal magneto-hydrodynamics are geodesic equations on the semidirect product of the group of volume preserving diffeomorphisms and the linear space of divergence free vector fields with $L^2$ metric [11] [12].

The Lagrangian (exponential) instability of geodesics is determined by the negativity of the sectional curvature. In this way Arnold [1] showed the instability in most directions of Euler equation for ideal flow, Shkoller [10] showed that averaged Euler equation is more stable than Euler equation for ideal flow, Misiolek [6] [13] has results on the stability of the Korteweg-de-Vries and Camassa-Holm equations and Zeitlin and Kambe [12] for ideal hydrodynamics.

In this paper we are concerned with the equations of motion of an ideal charged fluid in a given magnetic field $u_t = -\nabla_u u - \rho u \times B - \operatorname{grad} p$, $\rho_t = -d\rho.u$, $\operatorname{div} u = 0$ and the superconductivity equation (obtained for $\rho = 1$) as geodesic equations.

We consider the central extension by the 1-torus $\mathbb{T}$, due to Ismagilov [14][15], of the group of exact volume preserving diffeomorphisms of a compact Riemannian manifold $M$, corresponding to the Lichnerowicz Lie algebra cocycle $\omega(X, Y) = \int_M \eta(X, Y)\mu$, where $\eta$ is a closed 2-form on $M$. This extension exists only if the cohomology class of $\eta$ is integral. In the 3-dimensional case $\eta$ corresponds to the magnetic vector field $B$ and the integrability condition on $\eta$ is equivalent to flux quantization of magnetic field: $\int_S (B \cdot n)dS \in \mathbb{Z}$ for every closed surface $S$ in $M$. In the case $H^{n-1}(M, \mathbb{R}) = 0$, the group of exact volume preserving diffeomorphisms coincides with the group of volume preserving diffeomorphisms, and the superconductivity equation is proved to be the geodesic equation on the central extension with right invariant $L^2$-metric.

To drop out the condition $H^{n-1}(M, \mathbb{R}) = 0$ we can consider a non-central extension of the group of volume preserving diffeomorphisms: the group of volume preserving automorphisms of a principal $\mathbb{T}$-bundle $P$ over $M$ possessing a principal connection with curvature form $\eta$. Again we have to impose the integrality condition on $[\eta]$. We consider the right invariant $L^2$-metric coming from the Kaluza-Klein metric on $P$. Then the geodesic equations are the equations of motion of an ideal charged fluid. For $\rho = 1$ we get again the superconductivity equation as a geodesic equation.

To get information on the Lagrangian stability of the superconductivity equation, we compute the sectional curvature in the case of the flat 3-torus in both settings (general and central extension) and the result is: the superconductivity equation is more stable than Euler equation for ideal flow.

Like in the book of Arnold and Khesin [3], the approach followed throughout the paper is formal. For example a proof of the existence of geodesic flow on Fréchet manifolds is not available in the literature: for the intricacies of the rigorous geometric approach to this field see Hamilton's paper about the inverse function theorem of Nash and Moser [16]. The ILH (inverse limit

of Hilbert) Lie group setting and well posedness of the Cauchy problem for the superconductivity equation will be treated elsewhere.

## 2  Superconductivity equation

Let $M$ be an $n$-dimensional compact Riemannian manifold with Levi Civita connection $\nabla$ and volume form $\mu$. Let $B$ be an $(n-2)$ vector field on $M$ (i.e. $B \in C^\infty(\wedge^{n-2}TM)$) such that $\eta = (-1)^{n-2}i_B\mu$ is a closed two-form. The cross product of a vector field $X$ with $B$ is the vector field $X \times B = (i_{X \wedge B}\mu)^\sharp = (i_X\eta)^\sharp$.

Then we can write the generalized equations of motion of an ideal charged fluid for a time dependent divergence free vector field $u$ and a time dependent real smooth function $\rho$ on $M$:

$$u_t = -\nabla_u u - \rho u \times B - \operatorname{grad} p$$
$$\rho_t = -d\rho.u, \tag{1}$$

where the index $t$ denotes partial derivation by $t$. When at $t = 0$ the function $\rho$ is a constant, say $\rho = 1$, then $\rho$ is a constant at every $t$ and we get the generalized superconductivity equation

$$u_t = -\nabla_u u - u \times B - \operatorname{grad} p. \tag{2}$$

In case $M$ is a 3-dimensional manifold, $B$ is simply a vector field and the condition $\eta = -i_B\mu$ closed is equivalent to the condition $B$ divergence free. Moreover, the cross product is the usual vector product. Then equation (1) models the motion of an ideal charged fluid in a given magnetic field $B$: $u$ represents the velocity field and $\rho$ the charge density. The superconductivity equation (2) models the motion of a high density electronic gas in a magnetic field: $u$ represents the velocity field of the electronic gas. Its similarity to Euler equation comes from the fact that at high density, because of the repelling of particles, an electron gas behaves like a fluid.

Zeitlin [17] and Roger [18] already showed that the superconductivity equation can be regarded as the Euler equation on the central extension of the Lie algebra of divergence free vector fields on $M$ by the Lichnerowicz cocycle. Poisson structures for superconductors are given in [19].

If the closed 2-form $\eta$ is integral we will obtain both equations (1) and (2) as geodesic equations on extensions of the group of volume preserving diffeomorphisms. The integrality condition on $\eta$ translates into a quantization

of the magnetic flux: the flux of $B$ across any closed surface is integral

$$\int_S (B \cdot n) dS \in \mathbb{Z}. \tag{3}$$

This magnetic field seems to be produced by Dirac monopols.

# 3 Right invariant metrics on Lie groups

In this paragraph we give expressions for the geodesic equation, Levi-Civita covariant derivative and curvature for Lie groups with right invariant metrics (see [20] for a nice presentation of this subject).

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $\rho_x$ be the right translation by $x$. Any right invariant bounded Riemannian metric on $G$ is determined by its value at the identity $<,>: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, a positive definite bounded inner product on $\mathfrak{g}$. Let $g : I \to G$ be a smooth curve and $u : I \to \mathfrak{g}$ its right logarithmic derivative (the velocity field in the right trivialization) $u(t) = T\rho_{g(t)^{-1}}.g'(t)$. In terms of $u$ the geodesic equation for $g$ has the expression

$$u_t = -\operatorname{ad}(u)^\top u, \tag{4}$$

where $\operatorname{ad}(X)^\top$ is the adjoint of $\operatorname{ad}(X)$ with respect to $<,>$, if this adjoint does exist.

The right trivialization induces an isomorphism $R : C^\infty(G, \mathfrak{g}) \to \mathfrak{X}(G)$ given by $R_X(x) = T\rho_x.X(x)$. In terms of this isomorphism, the Levi-Civita covariant derivative is

$$\nabla_X^G Y = dY.R_X + \frac{1}{2}\operatorname{ad}(X)^\top Y + \frac{1}{2}\operatorname{ad}(Y)^\top X - \frac{1}{2}\operatorname{ad}(X)Y, \tag{5}$$

for $X, Y \in C^\infty(G, \mathfrak{g})$.

The sectional curvature $\mathcal{K}$ and the Riemannian curvature $\mathcal{R}$ are related by

$$\mathcal{K}(X, Y) = \frac{< \mathcal{R}(X, Y)Y, X >}{< X, X >< Y, Y > - < X, Y >^2}, \tag{6}$$

so the sign of the expression $< \mathcal{R}(X, Y)Y, X >$ determines the sign of the sectional curvature. Its expression in the right trivialization is

$$\begin{aligned}
< \mathcal{R}(X, Y)Y, X > = {} & \frac{1}{4}\|\operatorname{ad}(X)^\top Y + \operatorname{ad}(Y)^\top X\|^2 - \frac{3}{4}\|\operatorname{ad}(X)Y\|^2 \\
& - < \operatorname{ad}(X)^\top X, \operatorname{ad}(Y)^\top Y > \\
& - \frac{1}{2} < \operatorname{ad}(X)^\top Y, \operatorname{ad}(X)Y > - \frac{1}{2} < \operatorname{ad}(Y)^\top X, \operatorname{ad}(Y)X > .
\end{aligned} \tag{7}$$

4

**An example**: [1] [2] Let $G = \text{Diff}_{vol}(M)$ be the group of volume preserving diffeomorphisms of a compact Riemannian manifold $(M, g)$ with induced volume form $\mu$, Levi Civita covariant derivative $\nabla$ and Riemannian curvature tensor $R$. Let $\mathfrak{g} = \mathfrak{X}_{vol}(M)$ the Lie algebra of divergence free vector fields. We consider the right invariant metric on $G$ given by the $L^2$ inner product $< X, Y > = \int_M g(X, Y)\mu$. The transpose of $\text{ad}(X)$ is $\text{ad}(X)^\top Y = P(\nabla_X Y + (\nabla X)^\top Y)$, the geodesic equation in terms of the right logarithmic derivative is Euler equation for ideal flow

$$u_t = -\nabla_u u - \text{grad}\, p, \quad \text{div}\, u = 0, \tag{8}$$

the covariant derivative for right invariant vector fields is $\nabla_X^G Y = P\nabla_X Y$ and the curvature

$$< \mathcal{R}(X, Y)Y, X > = < R(X, Y)Y, X > \tag{9}$$
$$+ < Q\nabla_X X, Q\nabla_Y Y > - \|Q\nabla_X Y\|^2,$$

with $P$ and $Q$ the orthogonal projections on the spaces of divergence free respectively gradient vector fields.

Let $\mathbb{T}^2$ be the two-dimensional torus and $X_k$ the Hamiltonian vector field with Hamiltonian (stream function) $\cos kx$, $k \in \mathbb{Z}^2$, $x \in \mathbb{T}^2$. The curvature of $\text{Diff}_{vol}(\mathbb{T}^2)$ in any two-dimensional direction containing the direction $X_k$ is non-positive.

# 4 Geodesics on general Lie group extensions

Consider an exact sequence of homomorphisms of Lie groups

$$1 \to H \to E \to G \to 1 \tag{10}$$

and the corresponding exact sequence of homomorphisms of Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0, \tag{11}$$

i.e. the Lie group $E$ is an extension of $G$ by $H$. A section $s : \mathfrak{g} \to \mathfrak{h}$ induces the following mappings:

$$b : \mathfrak{g} \to \text{Der}(\mathfrak{h}), \quad b(X)Y = [s(X), Y] \tag{12}$$
$$\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}, \quad \omega(X_1, X_2) = [s(X_1), s(X_2)] - s([X_1, X_2]) \tag{13}$$

with properties

$$[b(X_1), b(X_2)] - b([X_1, X_2]) = \operatorname{ad}(\omega(X_1, X_2)) \tag{14}$$

$$\sum_{cycl} \omega([X_1, X_2], X_3) = \sum_{cycl} b(X_1)\omega(X_2, X_3). \tag{15}$$

The Lie algebra structure on the extension $\mathfrak{e}$, identified as vector space with $\mathfrak{g} \oplus \mathfrak{h}$ via the section $s$, in terms of $b$ and $\omega$ is [21]

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2] + b(X_1)Y_2 - b(X_2)Y_1 + \omega(X_1, X_2) \tag{16}$$

Special cases: when $\omega = 0$ we get semidirect products, when $\mathfrak{h}$ is abelian and $b = 0$ we get central extensions with Lie algebra cocycle $\omega$.

We consider also the right invariant metric on the Lie group $E$ given at the identity by the positive inner product on $\mathfrak{e}$:

$$< (X_1, Y_1), (X_2, Y_2) >_{\mathfrak{e}} = < X_1, X_2 >_{\mathfrak{g}} + < Y_1, Y_2 >_{\mathfrak{h}}, \tag{17}$$

where $<, >_{\mathfrak{g}}$ and $<, >_{\mathfrak{h}}$ are positive definite inner products on $\mathfrak{g}$ and $\mathfrak{h}$ such that the transposes $\operatorname{ad}(X)^{\top} : \mathfrak{g} \to \mathfrak{g}$ and $\operatorname{ad}(Y)^{\top} : \mathfrak{h} \to \mathfrak{h}$ exist for any $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. To write the transpose of $\operatorname{ad}(X, Y)$ in the extended Lie algebra, we have to impose further conditions. We suppose the transpose $b(X)^{\top} : \mathfrak{h} \to \mathfrak{h}$ exists for any $X \in \mathfrak{g}$ and there exist maps $h : \mathfrak{h} \to L(\mathfrak{g})$ linear (actually $h$ takes values in the space of skew-adjoint operators on $\mathfrak{g}$) and $l : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ bilinear, defined by the relations

$$< \omega(X_1, X_2), Y >_{\mathfrak{h}} = < h(Y)X_1, X_2 >_{\mathfrak{g}} \tag{18}$$

$$< b(X)Y_1, Y_2 >_{\mathfrak{h}} = < l(Y_1, Y_2), X >_{\mathfrak{g}} . \tag{19}$$

Then the transpose of $\operatorname{ad}(X, Y)$ is

$$\operatorname{ad}(X_1, Y_1)^{\top}(X_2, Y_2) = (\operatorname{ad}(X_1)^{\top} X_2 + h(Y_2)X_1 - l(Y_1, Y_2),$$
$$\operatorname{ad}(Y_1)^{\top} Y_2 + b(X_1)^{\top} Y_2) \tag{20}$$

**Proposition 1.** *With the conditions above, the geodesic equation on the Lie group extension $E$ with right invariant metric, written in terms of the right logarithmic derivative $(u, \rho)$ with $u : I \to \mathfrak{g}$ and $\rho : I \to \mathfrak{h}$ is:*

$$u_t = -\operatorname{ad}(u)^{\top} u - h(\rho)u + l(\rho, \rho)$$
$$\rho_t = -\operatorname{ad}(\rho)^{\top} \rho - b(u)^{\top} \rho \tag{21}$$

We are interested in the special case when $b(X)$ is skew-adjoint for all $X \in \mathfrak{g}$: $< b(X)Y_1, Y_2 > + < Y_1, b(X)Y_2 >= 0$. This will be called the isometric case, like in [22], where the semidirect product with this special condition is studied.

**Proposition 2.** *On an isometric extension of $\mathfrak{g}$ by abelian $\mathfrak{h}$, the geodesic equation in terms of the right logarithmic derivative $(u, \rho)$ is*

$$u_t = -\operatorname{ad}(u)^\top u - h(\rho)u$$
$$\rho_t = b(u)\rho \tag{22}$$

*and the curvature tensor at the identity for the extended group is:*

$$
\begin{aligned}
< \mathcal{R}^E((X_1, Y_1), (X_2, Y_2))(X_2, Y_2), (X_1, Y_1) >= &< \mathcal{R}^G(X_1, X_2)X_2, X_1 > \\
&- < b(X_1)Y_2 - b(X_2)Y_1, \omega(X_1, X_2) > -\frac{3}{4}\|\omega(X_1, X_2)\|^2 \\
&+ \frac{1}{4}\|h(Y_2)X_1 + h(Y_1)X_2\|^2 - < h(Y_1)X_1, h(Y_2)X_2 > \\
&+ < \omega(X_1, \nabla^G_{X_1} X_2), Y_2 > + < \omega(X_2, \nabla^G_{X_2} X_1), Y_1 > \\
&- < \omega(X_2, \nabla^G_{X_1} X_1), Y_2 > - < \omega(X_1, \nabla^G_{X_2} X_2), Y_1 >,
\end{aligned} \tag{23}
$$

*where $\mathcal{R}^G$ denotes the Riemannian curvature at the identity in $G$ and $\mathcal{R}^E$ in the extended group $E$.*

In the particular case of an isometric semidirect product of $\mathfrak{g}$ with an abelian Lie algebra $\mathfrak{h}$, the only non-zero term at the right hand side is $< \mathcal{R}^G(X_1, X_2)X_2, X_1 >$.

**Geodesics and curvature on one-dimensional extensions of Lie groups**

Next we consider another special case: one-dimensional central extensions $\tilde{G}$ of $G$ with Lie algebra $\tilde{g}$ defined by the Lie algebra cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Then the Lie algebra bracket on $\tilde{\mathfrak{g}}$ is $[(X_1, a_1), (X_2, a_2)] = ([X_1, X_2], \omega(X_1, X_2))$. On the extended group we consider the right invariant metric given at the identity by the following positive definite inner product on $\tilde{\mathfrak{g}}$: $< (X_1, a_1), (X_2, a_2) >=< X_1, X_2 > +a_1 a_2$.

Suppose the transpose $\operatorname{ad}(X)^\top : \mathfrak{g} \to \mathfrak{g}$ with respect to the inner product exists for all $X \in \mathfrak{g}$ and the Lie algebra cocycle can be written in the form $\omega(X_1, X_2) =< k(X_1), X_2 >$ with $k : \mathfrak{g} \to \mathfrak{g}$; it follows $k$ is skew-adjoint operator and $k(X) = h(1)X$.

**Proposition 3.** *With the conditions above, the geodesic equation on the one-dimensional central extension $\tilde{G}$, in terms of the right logarithmic derivative $(u, a)$ is*

$$u_t = -\operatorname{ad}(u)^\top u - ak(u), a \in \mathbb{R}. \qquad (24)$$

*The sign of the sectional curvature in the extended group is given by the sign of*

$$< \widetilde{\mathcal{R}}((X_1, a_1), (X_2, a_2))(X_2, a_2), (X_1, a_1) > = < \mathcal{R}(X_1, X_2)X_2, X_1 >$$
$$-\frac{3}{4}\omega(X_1, X_2)^2 + \frac{1}{4}\|k(a_1 X_2 - a_2 X_1)\|^2 \qquad (25)$$
$$-\omega(\nabla^G_{a_1 X_2 - a_2 X_1} X_1, X_2) + \omega(\nabla^G_{a_1 X_2 - a_2 X_1} X_2, X_1),$$

*where $\mathcal{R}$ denotes the Riemannian curvature at the identity in $G$ and $\widetilde{\mathcal{R}}$ in the extended group $\tilde{G}$.*

With this formula applied to the Virasoro-Bott extension, considering $L^2$ and $H^1$ metrics, we can recover the results of Misiolek on the sign of the sectional curvature and so on the Lagrangian (exponential) stability of the Korteweg-de-Vries and Camassa-Holm equations [6] [13].

## 5  Lichnerowicz cocycle

Let $M$ be a compact Riemannian manifold with Levi Civita connection $\nabla$ and volume form $\mu$. Any closed two-form $\eta$ on $M$ determines a Lie algebra cocycle, called Lichnerowicz cocycle, on the Lie algebra of divergence free vector fields $\mathfrak{X}_{vol}(M)$:

$$\omega(X, Y) = \int_M \eta(X, Y)\mu \qquad (26)$$

Does it exist a corresponding Lie group cocycle on the group of volume preserving diffeomorphisms $\operatorname{Diff}_{vol}(M)$? The existence of a smooth local group cocycle $c$ integrating $\omega$ follows from [23], where this result is proved in general for Lie groups modeled over sequentially complete locally convex spaces.

When the cohomology class of $\eta$ is integral, results of Ismagilov [14] [15] imply the existence of a global group extension of the subgroup of exact volume preserving diffeomorphisms $\operatorname{Diff}_{exact}(M)$ by the torus $\mathbb{T}$, corresponding to a Lichnerowicz Lie algebra extension determined by $\eta$. A volume preserving diffeomorphism is called exact if there is an isotopy $\varphi_t \in \operatorname{Diff}_{vol}(M)$

from $\varphi_0 = \mathrm{id}_M$ to $\varphi_1 = \varphi$ such that $\frac{d}{dt}\varphi_t = X_t \circ \varphi_t$ and $i(X_t)\mu$ is an exact $(n-1)$-form. In case $H^{n-1}(M, \mathbb{R}) = 0$, the subgroup of exact diffeomorphisms coincides with the identity component of $\mathrm{Diff}_{vol}(M)$.

**Proposition 4.** *Let $M$ be a compact manifold with $H^{n-1}(M, \mathbb{R}) = 0$ and $[\eta] \in H^2(M, \mathbb{Z})$. Then the generalized superconductivity equation*

$$u_t = -\nabla_u u - u \times B - \mathrm{grad}\, p. \tag{27}$$

*is a geodesic equation on a central extension of the group of volume preserving diffeomorphisms with right invariant $L^2$ metric.*

*Proof.* The transpose of $\mathrm{ad}(X)$ with respect to the $L^2$ inner product on the Lie algebra of divergence free vector fields exists: $\mathrm{ad}(X)^\top X = P\nabla_X X$. and the Lichnerowicz cocycle is of the required form $\omega(X, Y) = < k(X), Y >$ with $k(X) = P(X \times B)$. Then proposition 3 can be applied with initial condition $a = 1$. $\qquad\square$

# 6 Volume preserving automorphisms of a principal $\mathbb{T}$-bundle

Let $M$ be an $n$-dimensional Riemannian manifold with volume form $\mu$ and $\eta$ a closed differential two-form with integral cohomology class $[\eta]$. Then there exists a principal $\mathbb{T}$-bundle $\pi : P \to M$ and a principal connection form $\alpha \in \Omega^1(P)$ having curvature $\eta$. The associated Kaluza-Klein metric on $P$, defined at a point $x \in P$ by

$$\kappa_x(\tilde{X}, \tilde{Y}) = g_{\pi(x)}(T_x\pi.\tilde{X}, T_x\pi.\tilde{Y}) + \alpha_x(\tilde{X})\alpha_x(\tilde{Y}), \tag{28}$$

determines the volume form $\tilde{\mu} = \pi^*\mu \wedge \alpha$ on $P$.

The group of volume preserving automorphisms of the principal bundle $P$, denoted by $E = \mathrm{Diff}_{vol}(P)^\mathbb{T} = \mathrm{Diff}_{vol}(P) \cap \mathrm{Diff}(P)^\mathbb{T}$ is an extension of the group of volume preserving diffeomorphisms $G = \mathrm{Diff}_{vol}(M)$ by the abelian gauge group $H = C^\infty(M, \mathbb{T})$. The corresponding exact sequence of Lie algebras is

$$0 \to C^\infty(M, \mathbb{R}) \to \mathfrak{X}_{vol}(P)^\mathbb{T} \to \mathfrak{X}_{vol}(M) \to 0. \tag{29}$$

The principal connection $\alpha$ provides us with a section $s(X) = X^H$, $X^H$ representing the horizontal lift of the divergence free vector field $X$ on $M$. It is easy to verify that the horizontal lift $X^H$ is again divergence free, with

respect to the volume form $\tilde{\mu}$ on $P$. The identification of $\mathfrak{X}_{vol}(P)^{\mathbb{T}}$ with $\mathfrak{X}_{vol}(M) \oplus C^{\infty}(M, \mathbb{R})$ (as vector spaces only) via the section $s$ is $(X, f) \mapsto X^H + (f \circ \pi)\xi$, where $\xi$ denotes the fundamental vector field of the principal torus action (in particular $\alpha(\xi) = 1$).

The mapping $b$ induced by $s$ is minus the usual action of vector fields as derivations on the algebra of smooth functions $b(X)f = -df.X$ and $\omega$ is the restriction of the curvature two-form $\eta$ to divergence free vector fields, since the curvature measures the deviation of the horizontal bundle from integrability: $(\eta(X_1, X_2) \circ \pi)\xi = [X_1^H, X_2^H] - [X_1, X_2]^H$.

The $L^2$ inner product on $\mathfrak{e} = \mathfrak{X}_{vol}(P)^{\mathbb{T}}$ coming from the Kaluza-Klein metric on $P$ is easily shown to be the sum of the $L^2$ inner products on $\mathfrak{g} = \mathfrak{X}_{vol}(M)$ and $\mathfrak{h} = C^{\infty}(M, \mathbb{R})$:

$$< (X_1, f_1), (X_2, f_2) >_{\mathfrak{e}} = \int_M g(X_1, X_2)\mu + \int_M f_1 f_2 \mu \qquad (30)$$

(here we impose the condition that the integral of the 1-form $\alpha$ along the fibers of $P$ is 1; otherwise we get a constant factor at the right side of the equality above).

**Proposition 5.** *The geodesic equation on the group of divergence free automorphisms of the principal $\mathbb{T}$-bundle $P$ with right invariant $L^2$ metric coming from the Kaluza-Klein metric on $P$, written in terms of the right logarithmic derivative $(u, \rho)$, $u : I \to \mathfrak{X}_{vol}(M)$, $\rho : I \to C^{\infty}(M, \mathbb{R})$, is:*

$$u_t = -\nabla_u u - \rho u \times B - \operatorname{grad} p$$
$$\rho_t = -d\rho.u, \qquad (31)$$

*the equations of motion of an ideal charged fluid.*

*Proof.* All the conditions required in proposition 2 are fulfilled by this extension. The mapping $h : C^{\infty}(M, \mathbb{R}) \to L(\mathfrak{X}_{vol}(M))$ is $h(f)X = P(fX \times B)$ and the mapping $l : C^{\infty}(M, \mathbb{R}) \times C^{\infty}(M, \mathbb{R}) \to \mathfrak{X}_{vol}(M)$ is $l(f_1, f_2) = P(f_1 \operatorname{grad} f_2)$. We have again denoted by $P$ the orthogonal projection on the space of divergence free vector fields on $M$. The transpose $\operatorname{ad}(X)^{\top}X = P(\nabla_X X)$ and $b(X)$ is skew-adjoint. Now proposition 2 can be applied. $\square$

The geodesic equation on $\operatorname{Diff}_{vol}(P)^{\mathbb{T}}$, for the initial condition $(u_0, a)$ with $u_0 \in \mathfrak{X}_{vol}(M)$ and $a \in \mathbb{R}$, is $u_t = -\nabla_u u - au \times B - \operatorname{grad} p$, like for the central extension corresponding to the Lichnerowicz cocycle. So, if $a = 1$, we obtain the superconductivity equation as a geodesic equation also for arbitrary compact Riemannian manifolds $M$ (i.e. without the condition

10

$H^{n-1}(M, \mathbb{R}) = 0$ we had to impose in section 5), especially in the interesting case of the flat 3-torus $\mathbb{T}^3$. We remark that the vector subspace $\{(X, a) : X \in \mathfrak{X}_{vol}(M), a \in \mathbb{R}\}$ is not a Lie subalgebra of $\mathfrak{X}_{vol}(P)^{\mathbb{T}}$.

## 7   Curvature computations

The expressions in propositions 2 and 3 determine the sign of the sectional curvature and we get informations on the Lagrangian stability of geodesics. A geodesic is Lagrangian stable if all geodesics with sufficiently close initial conditions at time zero remain close for all $t \geq 0$ [24]. This is a notion of stability referring to the motion, not to the velocity field.

The sign of the sectional curvatures of the same two-dimensional plane spanned by $(X_1, a_1)$, $(X_2, a_2)$ with $a_1, a_2 \in \mathbb{R}$ can differ for the two extensions of $\text{Diff}_{vol}(M)$: the central extension $\tilde{G}$ and the general extension $E = \text{Diff}_{vol}(P)^{\mathbb{T}}$. Using $h(a)X = ak(X)$ for $a \in \mathbb{R}$ and $X \in \mathfrak{X}_{vol}(M)$ we get

$$< \mathcal{R}^E((X_1, a_1), (X_2, a_2))(X_2, a_2), (X_1, a_1) > = < \mathcal{R}^G(X_1, X_2)X_2, X_1 >$$
$$- \frac{3}{4}\|\eta(X_1, X_2)\|^2 + \frac{1}{4}\|k(a_1 X_2 - a_2 X_1)\|^2 \qquad (32)$$
$$- \omega(\nabla^G_{a_1 X_2 - a_2 X_1} X_1, X_2) + \omega(\nabla^G_{a_1 X_2 - a_2 X_1} X_2, X_1),$$

where $\nabla^G_X Y = P\nabla_X Y$ and the formula for $\mathcal{R}^G$ is (9). This expression is very similar to the expression of the curvature in the central extension, only instead of $\omega(X_1, X_2)^2 = (\int_M \eta(X_1, X_2)\mu)^2$ we get in this case $\|\eta(X_1, X_2)\|^2 = \int_M \eta(X_1, X_2)^2 \mu$.

**The flat 3-torus**

Next we do curvature computations for $M = \mathbb{T}^3$, the 3-torus, in both settings: central extension by $\mathbb{R}$ and general extension by $C^\infty(M, \mathbb{R})$, although in the first case only the existence of a local group cocycle integrating the Lichnerowicz cocycle is known ($H^2(\mathbb{T}^3) \neq 0$).

Let $M = \mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$ be the flat 3-torus. The Fourier basis is $e_k(x) = e^{ik\cdot x}$, where $k \in \mathbb{Z}^3$. We complexify the Lie algebra of divergence free vector fields on the torus, as well as the inner product, commutator and Levi Civita connection. A vector field on the torus is written in the Fourier basis as $u = \sum_k u_k e_k$ with $u_k \in \mathbb{C}^3$. The reality condition is $u_{-k} = \bar{u}_k$ and

the divergence free condition is $k \cdot u_k = 0$. Moreover

$$[u, v] = i((l \cdot u_k)v_l - (k \cdot u_l)v_k)e_{k+l} \tag{33}$$

$$Q(u) = \frac{1}{|k|^2}(k \cdot u_k)ke_k \tag{34}$$

$$\nabla_u v = i(l \cdot u_k)v_l e_{k+l}. \tag{35}$$

A 3-dimensional version of Arnold's non-positivity result for the curvature of the group of volume preserving diffeomorphisms on the 2-torus was obtained in [25]: for $U_p = u_p e_p + u_{-p}e_{-p}$ and $X = \sum_k v_k e_k$,

$$< \mathcal{R}(U_p, X)X, U_p >= -\|Q\nabla_{U_p}X\|^2 \leq 0. \tag{36}$$

**Proposition 6.** *Let $B = B_0$ be a constant vector field on the 3-torus. Let $U_p = u_p e_p + u_{-p}e_{-p}$ and $V_p = v_p e_p + v_{-p}e_{-p}$ be divergence free vector fields on the 3-torus with the additional condition $u_p$ and $v_p$ real and orthogonal. Then the sectional curvature of the two-dimensional plane spanned by $(U_p, 1)$ and $(V_p, 0)$ on the central extension $\tilde{G}$ (respectively on the general extension $E$) of $\mathrm{Diff}_{vol}(M)$ is positive if and only if $|u_p|^2 < \frac{1}{6(2\pi)^3}$ (respectively $|u_p|^2 < \frac{1}{3}$).*

*Proof.* We observe that $\nabla_{U_p}V_p = 0$, so in the curvature formula (14) applied to $(U_p, 1), (V_p, 0)$, only 2 terms are non-zero:

$$< \tilde{\mathcal{R}}((U_p, 1), (V_p, 0))(V_p, 0), (U_p, 1) >= -\frac{3}{4}\omega(U_p, V_p)^2 + \frac{1}{4}\|k(V_p)\|^2. \tag{37}$$

Using the Fourier coefficients, we get the following expressions of the cocycle $\omega$ and of the mapping $k$:

$$\omega(u, v)^2 = (\int_{\mathbb{T}^3} \mu(B, u, v)\mu)^2 = (2\pi)^6 (\sum \mu(B_k, u_l, v_m)\delta(k + l + m))^2 \tag{38}$$

$$\|\eta(u, v)\|^2 = \int_{\mathbb{T}^3} \mu(B, u, v)^2\mu = (2\pi)^3 \sum \mu(B_k, u_l, v_m)^2\delta(k + l + m) \tag{39}$$

and

$$k(u) = P(u \times B) = ((u_l \times B_k) - \frac{1}{|k + l|^2}((k + l) \cdot (u_l \times B_k))(k + l))e_{k+l}. \tag{40}$$

From the hypothesis, $(p, u_p, v_p)$ is an orthogonal basis of $\mathbb{R}^3$. In the associated orthonormal basis the coordinates of $B_0$ are denoted by $(a_1, a_2, a_3)$.

Let $\tilde{\mathcal{R}}$ (respectively $\mathcal{R}^E$) be the curvature tensor of the central (respectively general) extension of the group of volume preserving diffeomorphisms. Then

$$< \tilde{\mathcal{R}}((U_p, 1), (V_p, 0))(V_p, 0), (U_p, 1) > = -\frac{3}{4}(2\pi)^6(\mu(B_0, u_p, \bar{v}_p)$$
$$+ \mu(B_0, \bar{u}_p, v_p))^2 + \frac{1}{2}(2\pi)^3(B_0 \times v_p - \frac{1}{|p|^2}\mu(p, B_0, v_p)p)^2 \qquad (41)$$
$$= \frac{1}{2}(2\pi)^3 a_1^2 |v_p|^2 (1 - 6(2\pi)^3 |u_p|^2).$$

So the sectional curvature on the central extension is positive if and only if $|u_p|^2 < \frac{1}{6(2\pi)^3}$. An analogous computation for $\mathcal{R}^E$ gives the condition $|u_p|^2 < \frac{1}{3}$. $\qquad\square$

In conclusion, the superconductivity equation on the 3-torus is more stable than Euler equation for ideal flow.

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# References

[1]     V. Arnold, Ann. Inst. Fourier, 16 (1966) 319.

[2]     J. Marsden, D. Ebin and A. Fischer, Diffeomorphism groups, hydrodynamics and relativity, in: Proc. 13th Biennial Seminar of Canadian Math. Congress, ed. J.R.Vanstone, (Montreal, 1972).

[3]     V.I. Arnold and B.A. Khesin, Topological Methods in Hydrodynamics, Springer-Verlag 1998.

[4]     V.Y. Ovsienko and B.A. Khesin, Funct. Anal. Appl., 21 (1987) 329.

[5]     R. Camassa and D. Holm, Phys. Rev. Lett. 71 (1993) 1661.

[6]     G. Misiolek, Proc. Amer. Math. Soc., 125 (1997) 935.

[7]     S. Kouranbaeva, J. Math. Phys., 40 (1999) 857.

[8]     A. Constantin and H. McKean, Commm. Pure Appl. Math. 52 (1999) 949.

[9]     J.E. Marsden, T. Ratiu and S. Shkoller, The geometry and analysis of the averaged Euler equations and a new diffeomorphism group, submitted.

[10]    S. Shkoller, Journal of Functional Analysis, 160 (1998) 337.

[11]    J.E. Marsden, T.S. Ratiu and A. Weinstein, Trans. Am. Math. Soc. 281 (1984), 147.

[12]    V. Zeitlin and T. Kambe, J. Phys. A, 26 (1993), 5025.

[13]    G. Misiolek, J. Geom. Phys., 24 (1998) 203.

[14]    R.S. Ismagilov, Izv. Math. Nauk., 4 (1980).

[15]    R.S. Ismagilov, Representations of infinite-dimensional groups, AMS Translations of mathematical monographs, vol.152, 1996.

[16]    R. Hamilton, Bull. Amer. Math. Soc., 7 (1982) 65.

[17]    V. Zeitlin, Phys. Lett. A, 164 (1992) 177 and Physica D, 49 (1991) 353.

[18]    C. Roger, Rep. Math. Phys., 35 (1995) 225.

[19]    D. Holm and B.A. Kupershmidt, Phys. Lett. A, 93 (1983) 177.

[20]    P.W. Michor and T. Ratiu, Journal of Lie Theorie, 8 (1998) 293.

[21]    D. Alekseevsky, P.W. Michor and W. Ruppert, Preprint ESI 881 (2000).

[22]    C. Vizman, Geodesics and curvature of semidirect product groups, submitted.

[23]    K.-H. Neeb, Central extensions of infinite-dimensional Lie groups, submitted.

[24]    G. Misiolek, Indiana Univ. Math. Jour., 42 (1993) 215.

[25]    F. Nakamura, Y. Hattori and T. Kambe, J. Phys. A, 25 (1992).